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A Vector Heterogeneous Autoregressive Index Model for Realized Volatility Measures*

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Abstract

This paper introduces a new modelling for detecting the presence of commonalities in a set of realized volatility measures. In particular, we propose a multivariate generalization of the heterogeneous autoregressive model (HAR) that is endowed with a common index structure. The Vector Heterogeneous Autoregressive Index model has the property to generate a common index that preserves the same temporal cascade structure as in the HAR model, a feature that is not shared by other aggregation methods (e.g., principal components). The parameters of this model can be easily estimated by a proper switching algorithm that increases the Gaussian likelihood at each step. We illustrate our approach with an empirical analysis aiming at combining several realized volatility measures of the same equity index for three different markets.

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Keywords: Common volatility, vector heterogeneous autoregressive models, index models, combinations of realized volatilities.

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1 Introduction

The presence of co-movements in volatility measures is usually explained by a common reaction of investors, policy makers or central banks to news in some macroeconomic and financial variables. Engle and Marcucci (2006) find evidence in favor of the presence of common ARCH factors (Engle and Susmel, 1993) between 435 pairs obtained from 30 stocks of the Down Jones industrial index. However, their statistical approach might suffer from severe size distortions while applied in a multivariate setting (see Cubadda and Hecq, 2011; Hecq, Laurent and Palm, 2015). Anderson and Vahid (2007) propose to look at information criteria for determining the presence and the number of principal component factors out of 21 Australian weekly stock return volatilities. It turns out that this latter approach is probably more robust to the presence of jumps and fat tails than the canonical correlation framework of Engle and Marcucci (2006). However, in those contributions the dynamics of the system is assumed to be very parsimonious, in contradiction with the observed time series properties of daily volatility measures. For instance, the univariate Heterogeneous Autoregressive model (HAR, see Corsi, 2009) captures the long range dependence observed in daily time series by a restricted autoregressive model of order 22.

We propose in this paper a new modeling for analyzing the joint behavior of a set of daily volatility measures. First, we start with a multivariate version of the HAR, namely the Vector HAR (VHAR henceforth, see Bubák, Kočenda and Žikeš, 2011). Next, we test and consequently restrict the VHAR by means of a multivariate autoregressive index model (Reinsel, 1983). In particular, we impose proper reduced rank restrictions on the coefficient matrices of the VHAR to obtain the Vector Heterogeneous Autoregressive Index model (VHARI henceforth).

The VHARI is nested within the unrestricted VHAR, which is in turn a restricted version of a vector autoregressive model (VAR) of order 22. The VHARI provides a parsimonious modeling, whose forecasting performance can be compared with those of either less restricted multivariate models (e.g., VHAR or VAR(22)) or univariate HAR equations. At the representation theory level, the common factors obtained from the VHARI, namely the indexes, preserve the same temporal cascade structure as in the HAR, i.e., the weekly (monthly) index is equal to the weekly (monthly) moving average of the daily index. This is an important property of the VHARI that is not shared by most of the alternative aggregation methods (e.g., principal components, canonical correlations, etc.). Moreover, in a VHARI with one common component, a specification that is not rejected by the data in our empirical section of this paper, the unique index is generated by an univariate HAR model. This is also not generally the case for alternative aggregation strategies.

The rest of the paper is as follows. Section 2 presents the VHAR and the VHARI models as well as their implications. Statistical inference is discussed in details in Section 3. Note that we use a switching algorithm to maximize the Gaussian likelihood of a given VHARI specification. Hence, in principle the adequacy of our set of restrictions can be checked using either information criteria or likelihood ratio tests. This strategy cannot be implemented, for instance, for factors obtained through principal component analysis. Moreover, in the same vein as Takeuchi (1976), we propose some modified versions of the usual information criteria that are better suited for non-Gaussian series. In Section 4, a Monte Carlo simulation exercise documents the small sample properties of our modelling strategy. Section 5 applies the suggested framework in order to combine ten realized volatility measures of the same equity index for three different markets using data from the Oxford-Man Institute of Quantitative Finance. Finally, Section 6 concludes.

2 Model representation

2.1 The Vector Heterogeneous Autoregressive model

Our starting point for capturing the dynamic interactions within a set of n daily realized volatility measures $Y_t^{(d)} \equiv \left(Y_{1,t}^{(d)}, \dots, Y_{n,t}^{(d)}\right)'$ is a multivariate version of the univariate HAR model (Corsi, 2009) as used, inter alia, in Bubák et al. (2011), and Souček and Todorova (2013).

The vector $Y_t^{(d)}$ can include either the same kind of volatility measure (e.g., the realized variance)[1] for different markets in a study of volatility co-movements or several volatility measures (realized variance, bipower variation, etc.)[2] for the same market in order to construct an optimal linear combination like in Patton and Sheppard (2009). The latter analysis is pursued in Section 5 of this paper.

The Vector Heterogeneous Autoregressive model (VHAR) reads as follows:

$$Y_t^{(d)} = \beta_0 + \Phi^{(d)} Y_{t-1d}^{(d)} + \Phi^{(w)} Y_{t-1d}^{(w)} + \Phi^{(m)} Y_{t-1d}^{(m)} + \varepsilon_t, \quad t = 1, 2, ..., T, \tag{1}$$

where (d), (w), and (m) denote, respectively, time horizons of one day, one week (5 days a week), and one month (assuming 22 days within a month) such that

$$Y_t^{(w)} = \frac{1}{5} \sum_{j=0}^{4} Y_{t-jd}^{(d)}, \quad Y_t^{(m)} = \frac{1}{22} \sum_{j=0}^{21} Y_{t-jd}^{(d)}.$$

Innovations ε_t are i.i.d. with $\mathrm{E}(\varepsilon_t) = 0$, $\mathrm{E}(\varepsilon_t \varepsilon_t') = \Sigma$ (positive definite), finite fourth moments.

Beyond the fact that the HAR is a popular forecasting tool, two considerations arising from our empirical analysis have lead us to refer to (1) as a starting point. First, having estimated unrestricted VAR(p) models on a set of different volatility measures for each of the markets at hand, it emerges that we reject the null of no error autocorrelation for lags p equal to 5 or higher. This means that a higher dependence from the past is present in the data. In principle one could considerably increase the VAR order but the curse of dimensionality remains a problem even when the sample size is as large as in typical financial applications. Hence, (1) is a good compromise in terms of parameter proliferation since a VAR(22) has $N^2 \times 22$ mean parameters, whereas model (1) needs $N^2 \times 3$ of them. Second, for the considered set of realized volatilities the coefficient matrices

[1] Realized covariances may also be included in $Y_t^{(d)}$, see Fengler and Gisler (2015).

[2] The realized variances are computed using $RV_t \equiv \sum_{i=1}^{M} r_{t,i}^2$, where $r_{t,i}$ are the high frequency intra-day returns, observed for M intra-day periods each day. For instance $M = 79$ for 5-min returns when the market is open between 9 a.m. to 4 p.m. The bipower variation $BV_t \equiv \frac{\pi}{2} \frac{M}{M-1} \sum_{i=2}^{M} |r_{t,i}||r_{t,i-1}|$ is one of the measures of the integrated volatility that is designed to be robust to jumps. See, *i.a.*, Barndorff-Nielsen and Shephard (2004) and Bauwens, Hafner and Laurent (2012)

$\Phi^{(d)}, \Phi^{(w)}$ and $\Phi^{(m)}$ are far from being diagonals and consequently a set of individual HAR models does not seem appropriate.

Next subsection introduces additional meaningful restrictions to (1), namely the steps to go from the VHAR to the VHARI.

2.2 The VHAR-Index model

Let us further assume that (1) can be rewritten as follows

$$Y_t^{(d)} = \beta_0 + \beta^{(d)}\omega' Y_{t-1d}^{(d)} + \beta^{(w)}\omega' Y_{t-1d}^{(w)} + \beta^{(m)}\omega' Y_{t-1d}^{(m)} + \varepsilon_t, \tag{2}$$

where ω is a $n \times q$ full-rank matrix. In terms of parameters, (2) needs $4(n \times q) - q^2$ instead of $n^2 \times 3$ in (1). Following Reinsel (1983), we label (2) as the VHAR-index (VHARI) model. To some extent, the VHARI modeling is related to the pure variance model of Engle and Marcucci (2006) in the sense that a reduced-rank restriction is imposed to the mean parameters of a multivariate volatility model. However, a fundamental difference between (2) and the common volatility model (see also Hecq, Laurent and Palm, 2015) stems from the fact that the former has in general a different left null space for the loading matrices of the factors $\beta = \left[\beta^{(d)} : \beta^{(w)} : \beta^{(m)}\right]$. Obviously, common volatility is allowed in the VHARI model if there exists a full-rank $n \times s$ (with $s < q$) matrix such that $\delta'\beta = 0$.

Beyond the important aspect in terms of parsimony, there are two further motivations for using (2). First, the indexes $f_t^{(d)} = \omega' Y_{t-1d}^{(d)}$ obtained from (2) satisfy the property

$$f_t^{(w)} = \frac{1}{5}\sum_{j=0}^{4} f_{t-jd}^{(d)}, \quad f_t^{(m)} = \frac{1}{22}\sum_{j=0}^{21} f_{t-jd}^{(d)}. \tag{3}$$

as for the observed univariate realized volatilities. Hence, the temporal cascade structure of the HAR model is preserved meaning that the weekly (monthly) index is equal to the weekly (monthly) moving average of the daily index. This would not be generally the case with either traditional reduced-rank regression models as in Engle and Marcucci (2006) or principal component methods.

Second, premultiplying both side (2) by ω' yields

$$f_t^{(d)} = \omega'\beta_0 + \omega'\beta^{(d)} f_{t-1d}^{(d)} + \omega'\beta^{(w)} f_{t-1d}^{(w)} + \omega'\beta^{(m)} f_{t-1d}^{(m)} + \omega'\varepsilon_t, \tag{4}$$

which shows that the indexes themselves follow a VHAR model. When $q = 1$ the unique index is generated by an univariate HAR model. This property is not shared by alternative methods to aggregate time series (e.g., averages, principal components, canonical correlations, etc.) since the resulting linear combination would generally follow a rather complicated ARMA structure; see Cubadda, Hecq and Palm (2009), Hecq *et al.* (2015) and the references related to the final equation representation of multivariate models therein.

3 Statistical inference

In order to estimate the parameters of model (2), we resort to a switching algorithm that is widely applied in cointegration analysis (see Boswijk and Doornik, 2004, and the references therein). The strategy consists in alternating between estimating ω for a given value of β and Σ, and estimating β and Σ for a given value of ω. In details, the procedure goes as follows:

1. Conditional to an (initial) estimate of the ω, estimate β and Σ by OLS on (2).

2. Premultiplying both the sides of (2) by $\Sigma^{-1/2}$ one obtains

$$\Sigma^{-1/2}Y_t^{(d)} = \Sigma^{-1/2}\beta^{(d)}\omega' Y_{t-1d}^{(d)} + \Sigma^{-1/2}\beta^{(w)}\omega' Y_{t-1d}^{(w)} + \Sigma^{-1/2}\beta^{(m)}\omega' Y_{t-1d}^{(m)} + \Sigma^{-1/2}\varepsilon_t.$$

   Applying the Vec operator to both sides of the above equation and using the property $\text{Vec}(ABC) = (C' \otimes A)\text{Vec}(B)$ one gets

$$\begin{aligned}\text{Vec}\left(\Sigma^{-1/2}Y_t^{(d)}\right) &= \left(Y_{t-1d}^{(d)\prime} \otimes \Sigma^{-1/2}\beta^{(d)}\right)\text{Vec}(\omega') + \left(Y_{t-1d}^{(w)\prime} \otimes \Sigma^{-1/2}\beta^{(w)}\right)\text{Vec}(\omega') \\ &+ \left(Y_{t-1d}^{(m)\prime} \otimes \Sigma^{-1/2}\beta^{(m)}\right)\text{Vec}(\omega') + \text{Vec}\left(\Sigma^{-1/2}\varepsilon_t\right), \end{aligned} \tag{5}$$

   from which we can finally estimate by OLS the ω coefficients conditional to the previously obtained estimates of the parameters β and Σ.

3. Switch between steps 1 and 2 till numerical convergence occurs.

As shown by Boswijk (1995), the proposed switching algorithm has the property to increase the Gaussian likelihood in each step. With respect to the Newton-Raphson method that was originally proposed by Reinsel (1983), the suggested switching algorithm has several advantages and one disadvantage. On the one hand, the switching algorithm *(i)* is computationally simpler; *(ii)* it does not require any normalization condition on the parameters ω; *(iii)* it can be easily modified to impose over-identifying restrictions on both β and ω. On the other hand it converges slower than Newton-type methods. Consequently, it is important to properly choose the initial values for the index weights ω. We suggest to resort to a canonical correlation analysis between $Y_t^{(d)}$ and $\left(Y_{t-1d}^{(d)} + Y_{t-1d}^{(w)} + Y_{t-1d}^{(m)}\right)$. The canonical coefficients of the latter variable provide the Gaussian ML estimators of elements of ω when $\beta^{(d)} = \beta^{(w)} = \beta^{(m)}$.

Note that a numerical stability problem may arise when the number of series is very large. A possible solution is to resort to a properly "regularized" estimate of the autocorrelation matrix function of series $Y_t^{(d)}$ instead on the natural one that is implicitly used in our procedure (see Bernardini and Cubadda (2015) for details).

In order to identify the number of factors q, one can use the usual information criteria proposed by Schwarz (BIC), Hannan-Quinn (HQIC) and Akaike (AIC). We propose some variants of them that are based on the theoretical framework developed by Takeuchi (1976). In short, this author extends the AIC by relaxing the strong assumption that the set of the candidate models includes the true model. This extension is relevant in our case for at least two reasons. First, HAR processes are generally seen as an approximation to long-memory processes (Corsi, 2009). Second, the residuals of HAR models are typically non-Gaussian and heteroskedastic (e.g., Corsi, Audrino and Reno, 2012; Corsi, Mittnik, Pigorsch and Pigorsch, 2008), whereas our switching algorithm aims at maximizing the Gaussian likelihood. In the Appendix, we develop a Takeuchi-type modification to the traditional information criteria for our VHARI models. We denote the modified criteria as MAIC, MHQIC, and MBIC.

4 Monte Carlo analysis

This section presents a Monte Carlo study with the aim to evaluating the finite sample performances of our method. The previous section has shown how to estimate ω using a switching algorithm for a fix number of factors q of the VHARI. We now investigate the relative merits of both the traditional and modified information criteria for model identification, estimation, and forecasting.

In the Monte Carlo design we simulate demeaned realized volatilities $\overline{Y}_t^{(d)} = Y_t^{(d)} - \mathrm{E}\left(Y_t^{(d)}\right)$ that are generated by the following model:

$$\overline{Y}_t^{(d)} = A\Delta^{(d)}A^{-1}\overline{Y}_{t-1d}^{(d)} + A\Delta^{(w)}A^{-1}\overline{Y}_{t-1d}^{(w)} + A\Delta^{(m)}A^{-1}\overline{Y}_{t-1d}^{(m)} + \varepsilon_t, \quad t = 1, 2, ..., T, \tag{6}$$

where A is a full-rank $n \times n$ matrix, $\Delta^{(d)}$ is a diagonal matrix with the first q diagonal elements drawn from a $U_n(0.36, 0.399)$ and the remaining elements equal to zero, $\Delta^{(w)}$ and $\Delta^{(m)}$ are two diagonal matrices with the first q diagonal elements drawn from a $U_n(0.28, 0.30)$ and the remaining elements equal to zero. Notice that Equation (6) implies that series $\overline{Y}_t^{(d)}$ are generated by a VHARI model with parameters

$$\beta^{(j)} = A\Delta_{\bullet q}^{(j)},\ \omega' = A_{q\bullet}^{-1},\ \text{for } j = d, w, m,$$

where $\Delta_{\bullet q}^{(j)}$ is the matrix formed by the first q columns of $\Delta^{(j)}$, and $A_{q\bullet}^{-1}$ is the matrix formed by the first q rows of A^{-1}. In order to reproduce the positive co-movements of the realized volatility measures that we observe in our data, the matrix A is generated by a $n \times n$ half-normal distribution

In order to take into account the positiveness of realized variances as well as to reproduce the well known volatility in volatility phenomenon (Corsi *et al.*, 2008), elements of ε_t have a conditional log-normal error distribution with GARCH variances (see e.g., Barndorff-Nielsen and Shephard (2002) on the use of the log-normal distribution for realized volatilities). In particular, $\varepsilon_{it} = z_{it}\sqrt{h_{it}}$ for $i = 1, \ldots, n$, where $z_{it} = \left[u_{it} - \mathrm{E}(u_{it})\right] / \sqrt{\mathrm{Var}(u_{it})}$, $\ln(u_{it})$ is the i-th element of an i.i.d. $N(0, \mathrm{I}_n)$, and $h_{it} = 0.01 + 0.25\varepsilon_{it-1}^2 + 0.74h_{it-1}$.

We generate $1000 + 200$ observations of each series where the first 100 points are used as burn-in-period, the central $T = 1000$ observations are used for estimation, and the final points are used to compute 100 one-step ahead forecast errors. We use both $n = 10$ and $n = 20$ systems with $q = 1, 2, 4$ factors. Our choices about q and T are guided by the features of the variables that we analyze in the empirical application.

We evaluate the merits of the six information criteria by means of three statistics over 1000 replications. First, the percentage with which the estimated number of factor $\widehat{q}$ is equal to the

Table 1: Monte Carlo results

		$n = 10$			$n = 20$		
q	IC	$\%(\widehat{q} = q)$	RFD	ARMSFE	$\%(\widehat{q} = q)$	RFD	ARMSFE
1	BIC	99.10	19.47	89.70	**100.00**	**10.50**	**79.61**
	HQIC	90.50	21.44	89.94	95.50	11.10	79.70
	AIC	17.80	51.22	93.30	19.40	34.73	83.18
	MBIC	**100.00*****	**19.30*****	**89.68****	99.90	10.51	79.62
	MHQIC	98.8***	19.58***	89.72***	99.50***	10.57***	79.62***
	MAIC	68.20***	28.88***	90.72***	65.4***	18.82***	80.80***
2	BIC	98.80	27.83	91.31	**100.00**	**14.55**	**79.98**
	HQIC	89.70	29.92	91.55	96.40	15.02	80.05
	AIC	19.20	56.29	94.45	18.70	37.18	83.65
	MBIC	**99.10**	**27.72**	91.59	99.90	14.56	80.04
	MHQIC	98.80***	27.81***	**91.29*****	99.00***	14.73***	80.01***
	MAIC	65.10***	37.44***	92.29***	61.70***	22.99***	81.18***
4	BIC	**99.30*****	**42.61*****	94.11***	98.90***	**22.15*****	82.24***
	HQIC	94.40	43.57	94.12	97.80	22.44	82.19
	AIC	27.70	62.88	96.02	21.80	41.35	84.97
	MBIC	86.10	43.17	96.87	91.30	22.32	83.18
	MHQIC	99.10***	42.71***	**94.05**	**99.00*****	22.29***	**82.17****
	MAIC	71.59***	49.24***	94.65***	55.50***	31.81***	83.46***

Note: $\%(\widehat{q}= q)$ is the percentage with which each information criterion IC detects the true number of factors in the VHARI model. RFD is the Frobenius distance between the estimated parameters and the true ones relative to the Frobenius distance of OLS. ARMSFE is the average mean square forecast errors relative to the n HAR univariate forecasts. The best result for each pair (n, q) is in bold. ** (***) indicates significance at the 5% (1%) level of the t-tests of equal ARMSFEs or RFDs and of the McNemar's test on the differences between $\%(\widehat{q}= q)$ of the methods identified by a given IC and its Takeuchi-type version.

true one q. Second, the average of the mean square forecast errors relative to the unrestricted VHAR forecasts (ARMSFE). Third, the Frobenius distance between the estimated mean VHARI parameters and the true ones relative to the Frobenius distance of the OLS estimates of the mean VHAR parameters (RFD). In order to asses the significance of the differences in performances between the traditional information criteria and the modified ones, we use the t-test for the null hypothesis that the differences between the ARMSFEs or the RFDs over the 1000 replications are centered on 0, and the McNemar's test for the null hypothesis that the probabilities of identifying the true number of factors are the same. The results are reported in Table 1.

We observe that the MBIC is the best criterion according to all the three statistics when $q = 1$, and it does significantly better than the BIC for $n = 10$. When $q = 4$, the best criteria are instead BIC and MHQIC. The performances of BIC, MBIC, and MHQIC are similar when $q = 2$. The AIC, its modified version, and the HQIC never perform best. Regarding the usefulness of our modifications to the traditional information criteria, AIC and HQIC perform uniformly worse

Table 2: Data used

Realized volatility measure	Acronym
Realized Variance (5 min.)	RV5
Realized Kernel	RK
Realized Variance (5 min. using 1 min. subsamples)	RV5_1
Realized Variance (10 min.)	RV10
Realized Variance (10 min. using 1 min. subsamples)	RV10_1
Bipower Variation (5 min.)	BV5
Bipower Variation (5 min. using 1 min. subsamples)	BV5_1
Median Truncated Realized Variance	MTRV
Realized Semivariance (5 min.)	RSV5
Realized Semivariance (5 min. using 1 min. subsamples)	RSV5_1

than their modified counterparts, whereas matters are more controversial for the BIC. Indeed, MBIC significantly improves over BIC when $n = 10$ and $q = 1$ but the viceversa is true when $q = 4$ regardless of the system dimension. Interestingly, the most accurate forecasts are very often matched with the highest percentages of correct model identification. This suggests that, in case that the best performing information criteria should provide conflicting results, an out-of-sample forecasting exercise provides valuable information on the choice of q. We will pursue this strategy in the empirical application.

5 Empirical application

In this section we illustrate our approach with the aim to detecting the existence of common components within ten realized volatility measures. Following Patton and Sheppard (2009), the main idea is to build linear combinations of different volatility indicators and to evaluate their merits through an out of sample forecasting exercise. In particular, the target variable should be an unbiased proxy for the unobserved quadratic variation, whereas the predictors are (linear combinations of the) lags of both the individual indicators and their linear combinations. As Patton and Sheppard (2009) and Liu *et al.* (2015), we use the daily squared open-to-close return $\tilde{r}_t^2$ as our target variable.

We consider daily series spanning the period 01/01/2000 to 10/29/2015 of the measures that are reported in Table 2 for three equity indexes: S&P500 for the U.S., FTSE 100 for the U.K. and the Nikkei 225 for Japan. These series are downloaded from Oxford-Man Institute of Quantitative Finance's webpage (see Heber, Lunde, Shephard and Sheppard, 2009).

Before carrying out the evaluation of the VHARI as an aggregation method, we first check the adequacy of the VHARI restrictions. In particular, we use a rolling window of 1000 observations to compute the h-step ahead direct forecast for $h = 1, 5, 22$ from the individual HAR's, the unrestricted

Table 3: ARMSFE and quartiles of the $\widehat{q}$ distribution - S&P500

Method/criterion	ARMSFE $h=1$	$h=5$	$h=22$	[Q_1 Q_2 Q_3]
VHARI/BIC	94.4	**82.9**	94.2	[3 4 7]
VHARI/HQIC	94.6	84.4	95.8	[6 7 8]
VHARI/AIC	94.2	84.1	95.4	[9 10 10]
VHARI/MBIC	**92.9**	85.6	93.4	[1 1 1]
VHARI/MHQIC	94.3	85.7	**90.5**	[1 1 2]
VHARI/MAIC	94.5	84.3	93.8	[5 6 7]
VHAR	94.2	84.2	95.4	

Note: h is the forecasting horizon. ARMSFE is the average of the mean square forecast errors relative to the HAR univariate forecasts. Q_i indicates the i-th quartile of the number of factors distribution. The best result for each h among the multivariate methods is denoted in bold.

VHAR and the VHARI where the number of indexes q is chosen according to both the usual information criteria and the Takeuchi-type modified ones. In Tables 3 to 5 we report both the sum of the mean square forecast errors relative to the HAR forecasts (ARMSFE) and the quartiles of the number of factors distribution that are obtained by the various information criteria.

The results indicate that the VHARI outperforms the univariate HARs for the S&P500 and the Nikkei when $h = 5, 22$, whereas in the cases of FTSE and the Nikkei when $h = 1$ the reverse applies. Interestingly, the VHARI is always superior to the unrestricted VHAR and the former performs often best when q is chosen according to the MBIC. Looking at the quartiles of the empirical distributions of the estimated q, we see that the best forecasting model is almost uniformly associated with a small number of factors, mostly $q = 1$. Overall, these empirical findings suggest that using the VHARI to build a single linear combination of the 10 volatility indicators is appropriate for the analysis that follows.

Next, we resort again to a rolling window of 1000 observations to compute the direct h-step ahead forecasts from the model

$$\tilde{r}^2_{t+h} = \alpha_{0,h} + \alpha_{1,h} f_t^{(d)} + \alpha_{2,h} f_t^{(w)} + \alpha_{3,h} f_t^{(m)} + \varepsilon_{t,h}, \tag{7}$$

where the indexes in (7) are scalar time series, and those obtained from the model

$$\tilde{r}^2_{t+h} = \alpha_{0,h,i} + \alpha_{1,h,i} Y_{i,t}^{(d)} + \alpha_{2,h,i} Y_{i,t}^{(w)} + \alpha_{3,h,i} Y_{i,t}^{(m)} + \varepsilon_{t,h,i},$$

for the $i = 1, ..., 10$ individual realized volatility measures.

For each rolling sample, we construct the weights ω of the indexes in (7) with three alternative methods: *(i)* the n elements of ω are all set equal to 1/10 (i.e., the index at each frequency is

Table 4: ARMSFE and quartiles of the $\widehat{q}$ distribution - FTSE

	ARMSFE			
Method/criterion	$h=1$	$h=5$	$h=22$	[Q_1 Q_2 Q_3]
VHARI/BIC	108.6	111.5	106.6	[3 4 4]
VHARI/HQIC	110.1	114.2	**105.1**	[5 6 6]
VHARI/AIC	108.9	113.8	108.2	[8 9 10]
VHARI/MBIC	**105.8**	**106.7**	105.8	[1 1 2]
VHARI/MHQIC	109.1	108.4	112.6	[1 3 4]
VHARI/MAIC	108.5	113.3	109.4	[4 4 5]
VHAR	108.8	114.0	107.8	
See the notes of Table 3.				

Table 5: ARMSFE and quartiles of the $\widehat{q}$ distribution - NIKKEI

	ARMSFE			
Method/criterion	$h=1$	$h=5$	$h=22$	[Q_1 Q_2 Q_3]
VHARI/BIC	112.7	98.9	92.7	[1 4 5]
VHARI/HQIC	113.7	98.3	94.3	[3 5 9]
VHARI/AIC	113.9	97.6	93.3	[6 8 10]
VHARI/MBIC	**110.0**	**93.8**	**86.8**	[1 1 1]
VHARI/MHQIC	113.8	98.4	90.8	[1 4 4]
VHARI/MAIC	113.5	98.1	93.0	[4 5 6]
VHAR	113.8	97.3	92.2	
See the notes of Table 3.				

the simple mean of the individual volatility measures at that frequency); *(ii)* ω is estimated by the first principal component of the ten daily indicators; *(iii)* ω is estimated through the VHARI with $q = 1$.

In Tables 6 to 8 we report the mean square forecast errors relative to the forecasts obtained by Model (7) with uniform index weights (RMSFE henceforth). We use the simple mean index model as the benchmark because Patton and Sheppard (2009) show that it is difficult to beat it in forecasting comparisons. We also report the results of the version of the Diebold and Mariano (1995) test by Harvey, Leybourne and Newbold (1997). In particular the null hypothesis that the MSFE of a given model is the same as the benchmark is tested against the alternative that the worst of the two models has a larger MSFE. The use of more sophisticated tools, as the model confidence set by Hansen Lunde and Nason (2011), is beyond the scope of this empirical illustration.

Overall, the results are as follows. For the S&P500, the VHARI indicator model does much better than models based on both individual indicators and aggregates when $h = 1$. The improvement over the benchmark is significant at the 10%. For higher forecasting horizons, the VHARI factor performs similarly as the mean factor and the best individual indicator, i.e., the 5-minute realized semivariance. For the FTSE, the VHARI indicator performs best when $h = 1, 22$, although the improvements over the benchmark are not significant. When $h = 5$ the best performer is the model based on the median truncated realized variance. For the Nikkei, the realized kernel measure is significantly superior to the benchmark when $h = 1$ whereas the models based on the 5-minute realized variance and the VHARI factor are respectively the best performer when $h = 5$ and $h = 22$.

Finally, we can conclude from this illustration that the VHARI factor model is the best performer in one third of the cases. It is very often superior to models based on the simple mean and the first principal component. Moreover, there is no individual indicators that performs systematically better than the competitors along different markets and for different forecast horizons. These findings suggest that it may be worthy to add the VHARI to the toolkit of multivariate realized volatility modelling.

6 Conclusions

In this paper we have proposed the VHARI model, a multivariate generalization of the HAR model by Corsi (2009), which allows for a parsimonious modelling of a vector of realized volatilities. In particular, the realized volatility measures can be explained as linear functions of few indexes, which preserve the same temporal cascade structure as the autoregressive terms of the univariate HAR model. The parameters of the VHARI model can be estimated by means of a switching algorithm that increases the Gaussian likelihood at each step. Finally, we have illustrated the practical value of the proposed methods by an empirical application to a ten realized volatility measures for the S&P500, FTSE and the Nikkei equity indexes

Table 6: RMSFE - S&P500

	RMSFE		
Index	$h = 1$	$h = 5$	$h = 22$
VHARI	**87.3***	100.5	99.5
PC	99.7	100.0	100.0
RV5	98.0	99.1	99.5
RK	99.5	99.9	99.9
RV5_1	100.3	100.3	100.3
RV10	95.6	**98.3**	99.6
RV10_1	100.9	100.5	99.9
BV5	104.9	101.5	100.5
BV5_1	102.1	101.0**	100.5
MTRV	102.0	102.1**	100.7
RSV5	94.1	99.3	**99.4**
RSV5_1	96.7**	99.0	100.0

Note: RMSFE is the mean square forecast error relative to the mean factor forecast. VHARI is the index produced by the proposed model with $q = 1$. PC is the first principal component of the ten measures of realized volatility. See Table 2 for the remaining acronyms. * (**) indicates significance at 10% (5%) of the Diebold-Mariano test of equal RMSFE of a model and the benchmark. The best result is in bold.

Table 7: RMSFE - FTSE

	RMSFE		
Index	$h = 1$	$h = 5$	$h = 22$
VHARI	**97.8**	100.0	**98.8**
PC	100.1*	100.0	100.0
RV5	100.3*	99.9	99.9
RK	99.6	99.8	99.9
RV5_1	99.7*	100.1	100.0
RV10	100.1	100.6	100.6
RV10_1	99.7	100.2	100.3
BV5	101.0	99.9	99.8
BV5_1	99.9	99.8	100.2**
MTRV	98.8	**98.4**	99.2
RSV5	101.6**	99.9	99.9
RSV5_1	100.8	100.4	100.2

See the notes of Table 6.

Table 8: RMSFE - NIKKEI

	RMSFE		
Index	$h = 1$	$h = 5$	$h = 22$
VHARI	103.6	100.8	**99.3**
PC	100.1	99.9	100.0
RV5	100.4	**99.4**	100.0
RK	**96.0****	99.8	100.1
RV5_1	97.1	99.5	99.9
RV10	102.8	101.7*	100.4
RV10_1	98.3	100.6	100.1
BV5	102.3	99.9	99.8
BV5_1	98.9	100.5	100.1
MTRV	96.3	101.8	99.6
RSV5	101.7	100.8	100.0
RSV5_1	100.2	101.2	100.1

See the notes of Table 6.

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7 Appendix

Assume that the candidate models have the form

$$Y_t = \theta' X_t + \varepsilon_t \tag{8}$$

where Y_t and X_t are respectively random variables of dimension n and k, θ is $n \times k$ coefficient matrix, and ε_t are i.i.d. $N(0, \Sigma)$ errors that are independently distributed from X_t, whereas the true model is

$$Y_t = \mathrm{E}(Y_t | X_{0,t}) + \epsilon_t$$

where $X_{0,t}$ is a random variable of dimension $k_0 \leq k$ such that its elements are a subset of those of X_t, and ϵ_t are i.i.d. non-normal errors with $\mathrm{E}(\epsilon_t) = 0$, $\mathrm{E}(\epsilon_t \epsilon_t') = \Sigma_0$, and finite fourth moments.

If Model (8) is estimated by Gaussian ML, the penalty term of the AIC is not fully appropriate because its derivation is based on the assumption that the set of the candidate models includes the true model, see e.g., Burnham and Anderson (2002). Takeuchi (1976) relaxed this assumption and obtained the following criterion

$$\mathrm{TIC} = \ln(\widehat{\Sigma}) + 2\widehat{\eta}/T$$

where

$$\widehat{\eta} = \sum_{t=1}^{T} \widehat{\varepsilon}_t' \widehat{\Sigma}^{-1} \widehat{\varepsilon}_t h_{tt} + \frac{1}{2} \left[T^{-1} \sum_{t=1}^{T} \left(\widehat{\varepsilon}_t' \widehat{\Sigma}^{-1} \widehat{\varepsilon} \right)^2 - n(n+2) \right] \tag{9}$$

$h_{tt} = X_t' (\sum_{i=1}^{t} X_t X_t')^{-1} X_t$, $\widehat{\varepsilon}_t$ are the OLS residuals, and $\widehat{\Sigma}$ is the residual covariance matrix, see Yanagihara (2006) for further details.

Notice that if ϵ_t are i.i.d. Gaussian errors and the sample size T is large, the first term on the right-hand side of Equation (9) will be centered on nk,[3] whereas the second term will be centered on 0.[4] Hence, $\widehat{\eta} \simeq nk$, namely the number of free parameters in θ, as in the AIC.

However, when applying the Takeuchi's framework to identify the number of indexes q in the VHARI model (2), it is necessary to take into account that the number of free mean parameters is equal to $4nq - q^2$ instead of $3nq$, which would be the large sample mean of $\widehat{\eta}$ under Gaussianity. Hence, we propose to use

$$\widetilde{\eta} = \widehat{\eta} + q(n - q)$$

in place of $4nq - q^2$ in the formulae of the traditional information criteria. This leads to define the

[3] This result follows by using the law of iterated expectations and noting that $\sum_{t=1}^{T} h_{tt}$ is the trace of the projection matrix of variables X 's.

[4] This results follows from the fact that the term in square brackets in (9) is a consistent estimator of multivariate kurtosis.

following modified information criteria

$$\begin{aligned}
\mathrm{MAIC} &= \ln(\widehat{\Sigma}) + 2\widetilde{\eta}/T, \\
\mathrm{MHQIC} &= \ln(\widehat{\Sigma}) + 2\widetilde{\eta}\ln(\ln(T))/T, \\
\mathrm{MBIC} &= \ln(\widehat{\Sigma}) + \widetilde{\eta}\ln(T)/T,
\end{aligned}$$

which are robust to the presence of heteroskedasticity and excess kurtosis.